Images of the critical points of nonlinear maps

Roderick V. Jensen and Christopher R. Myers
Department of Applied Physics, Yale University, New Haven, Connecticut 06520
(Received 19 February 1985)

An explanation of the peaks of probability distributions of chaotic nonlinear maps is given, as well as an algorithm for calculating the locations of those peaks. This analysis also provides insights into the nature of periodic and aperiodic behavior in nonlinear maps.

I. INTRODUCTION

Much attention has recently been given to the properties of one-dimensional nonlinear maps of the form $x_{n+1} = f(x_n)$. In this paper we will consider the much-studied logistic map given by the equation

$$x_{n+1} = a x_n (1 - x_n) , \tag{1}$$

for $0 \leq a \leq 4$, $x \in [0,1]$. In particular, we will examine the images of the extrema, or critical points, of $f(x_n)$, which we will call the boundaries of the map. Analysis of boundaries helps to explain many diverse aspects of nonlinear maps, especially probability distributions in chaotic regions, the emergence of periodic orbits in such regions, and the effects of crises.[1] This description will enable us to better understand both the deterministic and probabilistic natures of the dynamics of the map.

An examination of these boundaries was initially motivated in an attempt to explain peaks in the probability distributions for nonlinear maps exhibiting chaotic behavior. In the logistic map, for $a > 3.57$, there are many values of a which seem to give rise to chaotic behavior. A histogram of many iterations for such an a would reveal a probability-distribution spread over many values of x, with peaks at certain values. As a varies, so do the locations of the peaks. In a bifurcation diagram of the map (see Fig. 1), these peaks appear as dark streaks. We ask the following question: Why is the probability distribution peaked at certain values of x? The answer is that the images of the critical points of the function have singular probability distributions, which coincide with the regions of high number density found in the bifurcation diagram.

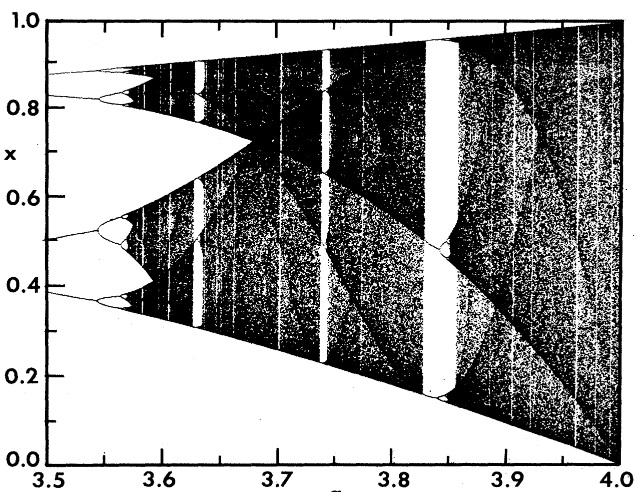

FIG. 1. The asymptotic orbits for the logistic map are plotted as a function of the control parameter a to form a bifurcation diagram. The relative density of points in the diagram reflects the structure of the corresponding invariant measure.

II. CONSTRUCTION OF BOUNDARIES

The return map defined by Eq. (1) is an inverted parabola, with a maximum at $x = 0.5$. For a given a, the function cannot attain a value greater than $a/4$. Therefore the long-time dynamics of the map are confined to the interval $[f(a/4), a/4]$. The first two images of the critical point delineate the exterior boundaries of the bifurcation diagram; the dynamics of the map do not fill the entire unit interval until $a = 4$. The subsequent images of the critical point are interior boundaries which correspond to the images of the critical points of the higher-order iterates of the logistic map.

The Nth iterate of the logistic map has $2N-1$ local extrema. One of these is always located at $x = 0.5$. The remaining points are grouped in $N-1$ pairs, each of which is a different order preimage of $x = 0.5$. Since the images of each element in the pairs are equal, there are a total of N distinct images of the $2N-1$ critical points. These N images generate $2N$ boundaries. These higher-order boundaries also delineate regions of the bifurcation diagram.

III. BOUNDARIES AND PROBABILITY DENSITY

In Fig. 2, we have plotted the first eight boundaries of the logistic map as a function of a for $3.5 \leq a \leq 4.0$, and superimposed this graph on a bifurcation plot of the same region. For $a < 3.57$, the accumulation point for the period-doubling bifurcations, the interior boundaries confine the periodic orbits. For $a > 3.57$, the boundaries not only confine the chaotic dynamics of the map, but also correspond exactly to the regions of high density described previously. The boundaries deviate from the bifurcation plot during periodic cycles, since the dynamics of such cycles are governed by fixed points and not boundaries. But in regions where the map is chaotic, the boundaries form a skeletal frame which gives shape to the map. We can now explain the probability distribution seen in the bifurcation diagram.

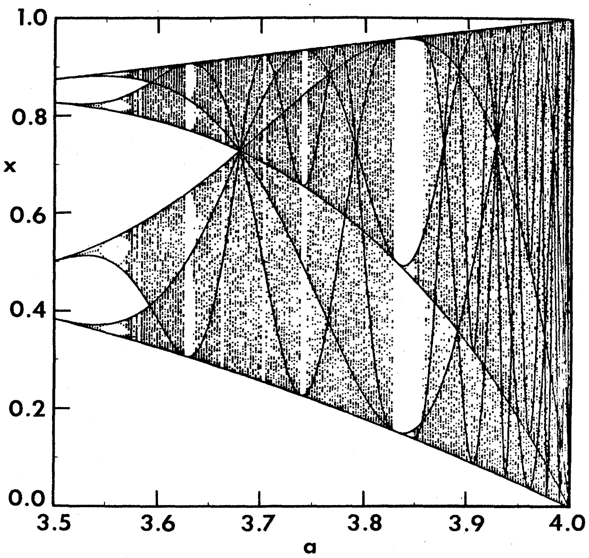

FIG. 2. The location of the first eight images of the critical point (boundaries) are plotted on the bifurcation diagram to show the correspondence with the peaks in the invariant probability distribution.

The number density of trajectories p landing in the neighborhood of some point x is given by the formula

$$p(x)=\sum_i \frac{p(x_i)}{|f'(x_i)|}, \tag{2}$$

where x_i denotes the preimages of the point x.[2] Neighborhoods with $|f'(x_i)|<1$ will be contracted upon iteration of the map, thus increasing the number density of the images of those neighborhoods. Conversely, neighborhoods with slope greater than one will be expanded, thereby decreasing number density. The critical point has a slope equaling zero, so the image of the critical point will have a singular distribution provided that the critical point has a nonzero probability density. If the return map behaves like $|x-x^*|^d$ in the neighborhood of the critical point x^*, then the singular probability distribution will behave like $1/|x-f(x^*)|^{(d-1)/d}$ in the vicinity of the boundary. In particular, for quadratic maps like the logistic map, $d=2$ and

$$p(x)\sim 1/\sqrt{|x-f(x^*)|}.$$

Guckenheimer[3] has proven that for a class of maps (which includes the logistic map) having no stable periodic orbit, the set of preimages of the critical point is dense. Thus the probability density of the critical point is nonzero. Successive iterations of the critical point will map the singularity from boundary to boundary, with a general decline in density, since in chaotic regions, the map has, on the average, an absolute slope greater than one, by definition of the (positive) Lyapunov exponent. If many boundaries come together for a given value of a, however, the density at that point may be greater than the densities at lower-order boundaries.

IV. INTERSECTIONS OF BOUNDARIES

The intersection of boundaries signifies the existence of a fixed point or periodic orbit at the point of intersection. Boundaries are successive images of the critical point(s), so if a boundary lands on a fixed point or periodic orbit, successive boundaries will also land on that point or orbit.

If the fixed point or orbit in question is stable, the boundaries join to form a stable perodic cycle. Boundaries coalesce and align along the cycle, interesecting at the superstable cycle. This is especially evident in Fig. 2 for $a\approx 3.83$, where the stable period-3 cycle appears. The realization that every stable orbit is characterized by this type of boundary coalescence enables one to identify stable cycles that might otherwise be too narrow to distinguish.

If the fixed point or orbit is unstable, however, the map is chaotic for that value of a. Singer[4] has proven that if a function with a single critical point has a stable periodic orbit, then the critical point will be attracted to it. At those points where boundaries intersect with an unstable fixed point the map does not have a stable periodic orbit. Misiurewicz[5] has built on this to prove that if a function has no stable periodic orbit and the critical point maps into an unstable fixed point, then the function has an absolutely continuous invariant measure, and is ergodic. And, finally, Jakobson[6] has proven that the set of parameter values for which the map has an absolutely continuous invariant measure has positive measure. Locating the unstable intersections of boundaries is a simple way to find chaotic behavior.

Two different behaviors of the map are possible at the unstable intersection of boundaries. One of these occurs when the intersection is imbedded within the bifurcation diagram, such as at $a\approx 3.79$ in Fig. 2. There the map is simply chaotic. If the intersection takes place at the edge of a region delineated by a boundary, however, a crisis occurs. A crisis has been defined by Grebogi, Ott, and Yorke as a collision between a chaotic attractor and a coexisting unstable fixed point or periodic orbit.[1] The theory of crises distinguishes between "boundary" crises and "interior" crises, the former bringing about a destruction of the chaotic attractor and its basin of attraction, and the latter causing a sudden change in the size of the attractor.

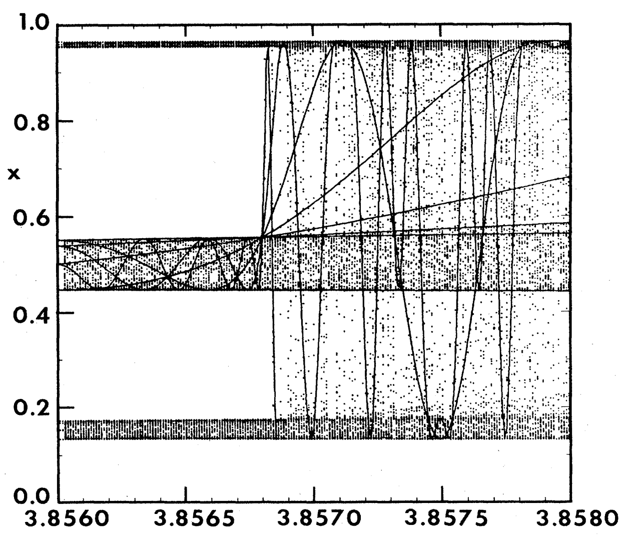

FIG. 3. The images of the critical points are plotted on the bifurcation diagram for values of a just after the period-3 cycle to illustrate the explosion of boundaries at a crisis point at $a \simeq 3.8568$.

To understand an interior crisis, consider the period-3 cycle that emerges at $a \approx 3.83$. This cycle undergoes period-doubling bifurcation, accumulating in chaos. Yet the dynamics of the map in this chaotic region beyond the accumulation point are constrained to three narrow bands, until the intersection of an unstable period-3 orbit with these bands brings about a crisis. Figure 3 is a magnification of the bifurcation/boundary diagram of Fig. 2, in the region of this crisis, with some boundaries omitted. Prior to the crisis, at $a = 3.8568$, higher-order boundaries are confined by the lower-order boundaries. At the point of intersection, however, all the interior boundaries break out of the regions that contained them. Lower-order boundaries diverge gradually from the confining regions, but higher-order boundaries, which oscillate rapidly, escape at very steep angles to the confining boundaries. These high-order boundaries, exploding out of the confining region at the point of crisis, are associated with the sudden widening of the attractor.

The same phenomenon occurs at the boundary crisis, for $a = 4$. All higher-order boundaries escape from the region of confinement, with lower-order boundaries escaping gradually, and higher-order curves racing away. But in this crisis, there are no remaining boundaries to confine the dynamics of the map, so the attractor becomes infinite, and is destroyed.

V. CONCLUSION

Examination of boundaries enables us to predict, by a simple algorithm, the regions of high probability density for nonlinear maps exhibiting stochastic behavior. It also allows us to predict which values of some controlling parameter will give rise to such behavior, and which will produce stable periodic cycles. Although this paper has examined only the logistic map, with a single extremum, this method of analysis also works for maps with multiple extrema, such as the circle map, given by the equation[7]

$$x_{n+1} = x_n + b \sin(2\pi x_n) + \tau \pmod 1 .$$

ACKNOWLEDGMENTS

The authors thank J. Yorke for providing the high-resolution bifurcation diagram displayed in Fig. 1. This work was supported by the National Science Foundation (under Grant No. PHY-83-08280).

[1] C. Grebogi, E. Ott, and J. Yorke, Physica D 7, 181 (1983).
[2] R. Shaw, Z. Naturforsch. Teil A **36**, 80 (1981).
[3] J. Guckenheimer, Commun. Math. Phys. **70**, 133 (1979).
[4] D. Singer, SIAM J. Appl. Math. **35**, 260 (1978).
[5] M. Misiurewicz, Publ. Math. Institut des Hautes Etudes Scientifiques (Bures-sur-Yvette, France) **52**, 17 (1980).
[6] M. V. Jakobson, Commun. Math. Phys. **81**, 39 (1981).
[7] L. Glass and R. Perez, Phys. Rev. Lett. **42**, 1772 (1982).